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COMMENT

A note on the gap between the first two eigenvalues for the Schrödinger operator

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Abstract. By means of a commutation formula, I give a simple proof of the upper bound of Wong *et al* on the gap between the first two eigenvalues in the Schrödinger operator. Unfortunately this proof does not seem to generalise into higher dimensions.

In the last two years there has been some interest in estimating the gap between the eigenvalues of the Schrödinger operator (Kirsch and Simon 1985, Wong *et al* 1984). The purpose of this comment is to give a simple proof of the upper bound on the gap between the first two eigenvalues found by Wong *et al* (1984). The proof I give here is based on a commutation formula and, unfortunately, only seems to work in one dimension.

Consider the one-dimensional Schrödinger equation

$$-\mathrm{d}^2u/\mathrm{d}x^2 + Vu = \lambda u \tag{1}$$

on the finite interval (a, b) with Dirichlet boundary conditions. Here $V(x)$ is a bounded non-negative potential defined on $[a, b]$. Let λ_1 and λ_2 denote the first and second non-zero eigenvalues of this equation. Then, we have the following.

Lemma.

$$\lambda_2 - \lambda_1 \leqslant 4\lambda_1. \tag{2}$$

Remarks.

(i) Actually Wong *et al* (1984) have proved this lemma in dimension n. In that case (2) reads $\lambda_2 - \lambda_1 \leqslant (4/n)\lambda_1$.

(ii) From equation (2) one can obtain the bound

$$\lambda_2 - \lambda_1 \leqslant [2\pi/(b-a)]^2 + 4 \sup_{[a,b]} V \tag{3}$$

(see Wong *et al* 1984).

Proof. Let u_1 be the ground state of the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V$ with Dirichlet boundary conditions on (a, b). Let v be the logarithmic derivative of u_1, i.e. $v = u_1'/u_1$ (here $'$ denotes differentiation with respect to x). It is well known (see, e.g., Crum

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1955, Deift 1978) that the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2+\tilde{V}$ with $\tilde{V}\equiv V-2v'$ has the same spectrum as $-\mathrm{d}^2/\mathrm{d}x^2+V$ except for λ_1. Thus, λ_2 is the ground state of $-\mathrm{d}^2/\mathrm{d}x^2+\tilde{V}$. We then estimate $\lambda_2-\lambda_1$ using the Rayleigh-Ritz variational principle. In fact,

$$\begin{aligned}\lambda_2-\lambda_1 &\leqslant [u_1, (-\mathrm{d}^2/\mathrm{d}x^2+\tilde{V})u_1]-[u_1, (-\mathrm{d}^2/\mathrm{d}x^2+V)u_1]\\ &=[u_1, (\tilde{V}-V)u_1]=-2(u_1, v'u_1)\\ &=-2\int_a^b u_1^2 v'\,\mathrm{d}x.\end{aligned}\tag{4}$$

Finally, integrating the right-hand side of (4) by parts we get

$$\lambda_2-\lambda_1\leqslant 4\int_a^b u_1u_1'v\,\mathrm{d}x=4\int_a^b (u_1')^2\,\mathrm{d}x.\tag{5}$$

However,

$$\lambda_1=\int_a^b (u_1')^2\,\mathrm{d}x+\int_a^b Vu_1^2\,\mathrm{d}x$$

and V is non-negative; hence $\int_a^b (u_1')^2\,\mathrm{d}x\leqslant\lambda_1$, which concludes the proof of the lemma.

Unfortunately, the commutation formula we use here is very much tied to one dimension and therefore this proof does not seem to go over into higher dimensions.

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